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Analytic resummation for the quark form factor in QCD

Lorenzo Magnea*

Dipartimento di Fisica Teorica, Università di Torino
and I.N.F.N., Sezione di Torino
Via P.Giuria 1, I–10125 Torino, Italy

Abstract

The quark form factor is known to exponentiate within the framework of dimensionally regularized perturbative QCD. The logarithm of the form factor is expressed in terms of integrals over the scale of the running coupling. I show that these integrals can be evaluated explicitly and expressed in terms of renormalization group invariant analytic functions of the coupling and of the space–time dimension, to any order in renormalized perturbation theory. Explicit expressions are given up two loops. To this order, all the infrared and collinear singularities in the logarithm of the form factor resum to a single pole in ϵ, whose residue is determined at one loop, plus powers of logarithms of ϵ. This behavior is conjectured to extend to all loops.

*e-mail: magnea@to.infn.it

1 Introduction

The asymptotic behavior of the electromagnetic form factor of charged particles at high energy has been the object of theoretical studies for almost half a century. It is the simplest gauge theory amplitude to exhibit a doubly logarithmic behavior, as a consequence of its infrared and collinear singularities. Thus the understanding of its asymptotic energy dependence requires a resummation of perturbation theory, going beyond renormalization group logarithms. This resummation was performed for the first time, at the leading logarithm level, by Sudakov [1], who condidered the off–shell form factor for an abelian gauge theory. He found that the leading (double) logarithms exponentiate, which results in the strong suppression of the elastic scattering of charged particles at high energy which bears his name. A similar exponentiation for the on–shell form factor was obtained in [2], and in later years the result was extended to nonabelian gauge groups [3].

Further extending the exponentiation to all subleading logarithms is a nontrivial task, which requires the complex machinery of perturbative factorization theorems. This goal was achieved by Mueller [4] and Collins [5] for abelian theories, and shortly later by Sen [6] for QCD. In essence, the complete exponentiation is achieved by showing that the energy dependence of the form factor must follow an evolution equation, which embodies the constraints of renormalization group and gauge invariance. This evolution equation was typically solved in terms of an undetermined initial condition, containing low–energy information and depending on some chosen infrared regulator, say a gluon mass. It was later realized [7] that dimensional regularization can be used directly at the level of the evolution equation. One can then give a simple and explicit solution for the form factor, using as initial condition the electric charge at vanishing momentum transfer. When the solution is written in this form, all infrared and collinear poles are explicitly exponentiated, and the results can be directly compared with diagrammatic calculations.

The quark form factor, although divergent, is a quantity of considerable interest for phenomenology: in fact, it appears in the computation of several cross sections, and techniques similar to the ones used for its exponentiation have been applied to several processes of interest in high energy physics, in particular to processes in which the perturbative series needs to be resummed because of logarithmic enhancements in special regions of phase space. Outstanding examples are the inclusive Drell–Yan cross section [8] and the transverse momentum distribution of

vector bosons produced in hadronic collisions [9]. More recently, similar techniques have been generalized to hadronic cross section with colored particles in the final state [10]. Strikingly, the quark form factor enters directly in the hard partonic cross section for the Drell–Yan process in the DIS scheme, which is proportional to the modulus squared of the ratio of the timelike to the spacelike form factor [8, 11]. In [7] it was shown that this ratio is given by an infinite phase (a generalization of the Coulomb phase, expressed by a series of pure counterterms in the $\overline{MS}$ scheme), times a manifestly finite exponential factor.

In the present paper, I will build upon the results of Ref. [7], and show that the resummed expression for the form factor can be explicitly evaluated to the desired accuracy, thus expressing the logarithm of the form factor as an analytic function of $\alpha_s(Q^2)$ and ϵ, manifestly renormalization group invariant. The key observation is the following: typically, in resummed expressions, integrals over the renormalization scale cannot be explicitly evaluated, because the one–loop running coupling has a Landau pole on the integration contour, which extends from the hard scale, $\mu^2 = Q^2$, all the way down to $\mu^2 = 0$, in the region dominated by nonperturbative effects. The sensitivity to the Landau pole, which can be estimated using various methods and models [12], is then taken as a measure of the influence of nonperturbative effects on the given observable. In dimensional regularization, however, the running coupling depends not only on the renormalization scale, but also on the dimensional regularization parameter ϵ. At one loop, the coupling still exhibits a Landau pole, but for general (complex) ϵ the pole is located at complex values of the renormalization scale μ^2, away from the integration contour. As a consequence, all integrals appearing in the logarithm of the form factor are well–defined, and can be evaluated. Alternatively, one can express the one–loop running coupling as a power series in terms of $\alpha_s(Q^2)$, and integrate the series term by term: one finds that there is no factorial growth in the coefficients of the series obtained after integration; in fact, the series can be summed to reconstruct the same analytic function of α_s and ϵ that was found by direct integration. It is easy to see how these results generalize when the running coupling is expressed including higher orders in the QCD β function. The complexity of the answer grows, but the most relevant features remain unchanged; in particular, the answer is manifestly invariant under changes in the renormalization scale, to the given accuracy in the β function; also, infrared and collinear poles resum, so that the logarithm of the form factor has only

a single pole in ϵ, as well as cut on the negative real axis in the ϵ plane; conventional perturbative results can be recovered by reexpanding in powers of α_s for finite ϵ.

The outline of the paper is the following: in Section 2, I briefly review the known results on the exponentiation of the quark form factor. In Section 3, I illustrate the power of dimensional regularization by solving explicitly to all orders the recursion relation arising from the renormalization group equation for the generalized Coulomb phase: all the towers of poles (leading, next–to–leading, etc.) arising in the series of counterterms can be expressed in terms of the perturbative coefficients of the relevant anomalous dimension, and of the QCD β function; all the resulting series can be summed, and they give functions of ϵ with at most a logarithmic singularity; the leading behavior as $\epsilon \to 0$ can be explicitly computed to all orders. In Section 4, I turn to the main issue, and compute the logarithm of the form factor using one–loop results only, as well as the one–loop running coupling; the result is remarkably simple, a single dilogarithm and a single logarithm of functions of α_s and ϵ, explicitly independent of the choice of renormalization scale. In Section 5, I generalize the calculation to two loops, obtaining again a relatively simple answer, containing only logarithms and dilogarithms. It is fairly clear that the calculation could be formally pushed to even higher orders, adding to the complexity of the answer, but without changing its basic features; furthermore, the perturbative coefficients of the various functions appearing in the logarithm of the form factor are known only to two loops. Section 6 summarizes the results and briefly discusses possible applications and future developments.

2 Exponentiation of the form factor

The techniques needed to compute the quark form factor in perturbative QCD to all orders in the logarithms of the energy are described in detail in Ref. [13]. Here I will just give the relevant definitions and sketch the main results, in the spirit of Ref. [7].

The (timelike) quark form factor in dimensionally regulated massless QCD is defined by

$$\begin{aligned} \Gamma_\mu(p_1, p_2; \mu^2, \epsilon) &= \langle 0 | J_\mu(0) | p_1, p_2 \rangle \\ &= -{\rm i} e e_q \, \overline{v}(p_2) \gamma_\mu u(p_1) \; \Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) \, , \end{aligned} \tag{2.1}$$

where $J_\mu(x)$ is the electromagnetic quark current, while $|p_1, p_2\rangle$ is a state containing a quark and an antiquark with momenta p_1 and p_2, and total energy $Q^2 = (p_1 + p_2)^2$, taken to be much larger than the confinement scale. In Eq. (2.1) I made use of the fact that in the massless limit the perturbative form factor has only one spin structure, and I defined the scalar form factor Γ to be normalized to 1 in the absence of strong interactions. Eq. (2.1) is written in the renormalized theory, so that all ultraviolet divergences arising in perturbation theory have been dealt with, and one can take $\epsilon = 2 - d/2 < 0$ to regulate infrared and collinear poles. Notice that, as a consequence of electromagnetic current conservation, the anomalous dimension of the form factor vanishes.

The conventional analysis of the form factor [13] proceeds through the steps that are characteristic of the derivation of perturbative factorization theorems: one starts by identifying the integration regions in momentum space that contribute to the form factor at leading power[1] of Q^2; next, one constructs a factorized expression for the form factor, in terms of (nonlocal) operators, typically defined in terms of eikonal lines, whose matrix elements reproduce the leading behavior in the various regions. Each operator factor will typically depend both on the gauge choice and on the renormalization scale; imposing gauge and renormalization group (RG) invariance leads to an evolution equation describing the energy dependence of the whole form factor. In dimensional regularization, this equation takes the form

$$Q^2 \frac{\partial}{\partial Q^2} \log \left[\Gamma \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] = \frac{1}{2} \left[K \left(\epsilon, \alpha_s(\mu^2) \right) + G \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] . \quad (2.2)$$

The functions K and G are characterized as follows: K is a pure counterterm, thus a series of poles in the $\overline{MS}$ scheme, which I will use throughout; G contains all the energy dependence, and is finite as $\epsilon \to 0$; the sum $K + G$ must be renormalization group invariant, because the form factor is; thus K and G renormalize additively, according to

$$\begin{aligned} \left(\mu \frac{\partial}{\partial \mu} + \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \right) G \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) & = \gamma_K(\alpha_s) \\ & = - \left(\mu \frac{\partial}{\partial \mu} + \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \right) K \left(\epsilon, \alpha_s \right) . \end{aligned} \quad (2.3)$$

[1] Technically, this is done using a massive regulator; in dimensional regularization *all* the energy dependence is logarithmic, leading regions are responsible for infrared and collinear poles, and one is in fact working with the *whole* form factor.

The anomalous dimension γ_K is well known in perturbative QCD. It is essentially the anomalous dimension associated with a cusp in a Wilson line, corresponding here to the hard vertex where the quark and the antiquark annihilate [14]. It is independent of ϵ, and can be expanded perturbatively as

$$\gamma_K(\alpha_s) = \sum_{n=1}^{\infty} \gamma_K^{(n)} \left(\frac{\alpha_s}{\pi}\right)^n \ , \tag{2.4}$$

with coefficients known to two loops [15]

$$\gamma_K^{(1)} = 2C_F \qquad ; \qquad \gamma_K^{(2)} = \left(\frac{67}{18} - \zeta(2)\right) C_A C_F - \frac{5}{9} n_f C_F \ . \tag{2.5}$$

Since we are dealing with divergent quantities, it is important to keep $\epsilon < 0$ throughout. Thus the β function is defined by

$$\begin{aligned} \beta(\epsilon, \alpha_s) &= \mu \frac{\partial \alpha_s}{\partial \mu} = -2\epsilon\alpha_s + \hat{\beta}(\alpha_s) \ , \\ \hat{\beta}(\alpha_s) &= -\frac{\alpha_s^2}{2\pi} \sum_{n=0}^{\infty} b_n \left(\frac{\alpha_s}{\pi}\right)^n \ . \end{aligned} \tag{2.6}$$

At the leading nontrivial order, the solution of Eq. (2.6) is

$$\overline{\alpha}\left(\frac{\mu^2}{\mu_0^2}, \alpha_s(\mu_0^2), \epsilon\right) = \alpha_s(\mu_0^2) \left[\left(\frac{\mu^2}{\mu_0^2}\right)^{\epsilon} - \frac{1}{\epsilon}\left(1 - \left(\frac{\mu^2}{\mu_0^2}\right)^{\epsilon}\right) \frac{b_0}{4\pi} \alpha_s(\mu_0^2)\right]^{-1} \ , \tag{2.7}$$

with $b_0 = (11C_A - 2n_f)/3$. Using this form of the running coupling, and its higher order generalizations, one can solve the renormalization group equation for the function G, as

$$\begin{aligned} G\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) &= G\left(-1, \overline{\alpha}\left(\frac{-Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right), \epsilon\right) \\ &+ \frac{1}{2} \int_{-Q^2}^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_K \left(\overline{\alpha}\left(\frac{\lambda^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right) \ , \end{aligned} \tag{2.8}$$

where the initial condition was chosen to emphasize that G is real for negative Q^2.

The second important consequence of dimensional regularization is the fact that it provides a natural initial condition for Eq. (2.2), in terms of which one can construct an explicit solution. In fact, one readily recognizes that each term in the perturbative expansion of $\Gamma(Q^2)$ must be proportional to a positive integer power of $(\mu^2/(-Q^2))^\epsilon$. Thus for fixed $\epsilon < 0$ all perturbative corrections to the form factor vanish in the limit $Q^2 \to 0$, and one can use

$$\Gamma\left(0, \alpha_s(\mu^2), \epsilon\right) = 1 \ . \tag{2.9}$$

One finds then

$$\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{-Q^2}\frac{d\xi^2}{\xi^2}\left[K\left(\epsilon,\alpha_s(\mu^2)\right)\right.\right. \tag{2.10}$$
$$\left.\left. + \; G\left(-1,\overline{\alpha}\left(\frac{\xi^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right),\epsilon\right)+\frac{1}{2}\int_{\xi^2}^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K\left(\overline{\alpha}\left(\frac{\lambda^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right)\right]\right\} .$$

Eq. (2.10) can be used directly as a generating function of diagrammatic results in dimensional regularization; for example, using one–loop results for the various functions appearing in the exponent, as well as the one–loop running coupling re–expanded in terms of $\alpha_s(Q^2)$, one reproduces correctly the quartic and cubic poles in ϵ at $\mathcal{O}(\alpha_s^2)$. Introducing the two–loop contribution to the anomalous dimension γ_K, one matches the double pole as well; the only singularity requiring a full two–loop computation (the knowledge of $G^{(2)}$, as well as $\mathcal{O}(\epsilon^2)$ terms in $G^{(1)}$), is the single pole, in agreement with the general considerations of Ref. [16]. In Ref. [7] it was shown that Eq. (2.10) can be used to evaluate explicitly the ratio of the timelike to the spacelike form factor, in terms of a contour integral in the complex plane of the renormalization scale ξ^2. The result (at $\mu^2 = Q^2$) is of the form

$$\frac{\Gamma\left(1,\alpha_s(Q^2),\epsilon\right)}{\Gamma\left(-1,\alpha_s(Q^2),\epsilon\right)} = \exp\left\{\mathrm{i}\frac{\pi}{2}K\left(\epsilon,\alpha_s(Q^2)\right)+\frac{\mathrm{i}}{2}\int_0^{\pi}d\theta\left[G\left(-1,\overline{\alpha}\left(\mathrm{e}^{\mathrm{i}\theta},\alpha_s(Q^2),\epsilon\right),\epsilon\right)\right.\right.$$
$$\left.\left. - \; \frac{\mathrm{i}}{2}\int_0^{\theta}d\phi\;\gamma_K\left(\overline{\alpha}\left(\mathrm{e}^{\mathrm{i}\phi},\alpha_s(Q^2),\epsilon\right)\right)\right]\right\} . \tag{2.11}$$

Eq. (2.11) proves that the ratio of form factors is given by an infinite phase times a finite exponential factor. The derivation of Eq. (2.11) also shows that the ratio is dominated by ultraviolet contributions: in fact, it is expressed by a contour integral that never approaches the origin. Thus, one may conclude that it is free of power corrections related to the Landau pole. In particular, the integrals in Eq. (2.11) can be evaluated explicitly to the desired accuracy in the running coupling; the result is of direct phenomenological relevance, since the modulus squared of the ratio of form factors appears in the resummed partonic Drell–Yan cross section [8, 17].

I will now concentrate on Eq. (2.10), and examine the structure of the exponent in more detail. In particular, I will be concerned with the fate of the Landau pole, that is expected to make the integration over the infrared region ill–defined. It will become apparent that dimensional regularization provides a solution to this problem. I will however start with an exercise in the application of the renormalization group, which displays in a simple context the power of dimensional regularization.

3 The counterterm function

Let us begin by considering the "Coulomb phase" $K(\epsilon, \alpha_s)$. In any minimal renormalization scheme, it is a series of poles which can be written as[2]

$$
\begin{aligned}
K(\epsilon, \alpha_s) &= \sum_{n=1}^{\infty} \frac{K_n(\alpha_s)}{\epsilon^n} \,, \\
K_n(\alpha_s) &= \sum_{m=n}^{\infty} K_n^{(m)} \left(\frac{\alpha_s}{\pi}\right)^m \,.
\end{aligned}
\tag{3.1}
$$

In such a scheme, $K(\epsilon, \alpha_s)$ has no explicit dependence on the renormalization scale μ. The renormalization group equation (2.3) then reduces to

$$
\beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} K\left(\epsilon, \alpha_s\right) = -\gamma_K(\alpha_s) \,,
\tag{3.2}
$$

which, given the finiteness of γ_K and the form of $\beta(\epsilon, \alpha_s)$, turns into a recursion relation for the perturbative coefficients of K. Explicitly

$$
\begin{aligned}
\alpha_s \frac{d}{d\alpha_s} K_1(\alpha_s) &= \frac{1}{2} \gamma_K(\alpha_s) \,, \\
\alpha_s \frac{d}{d\alpha_s} K_{n+1}(\alpha_s) &= \frac{1}{2} \hat{\beta}(\alpha_s) \frac{d}{d\alpha_s} K_n(\alpha_s) \,.
\end{aligned}
\tag{3.3}
$$

Eq. (3.3) expresses the fact that the physics of the function $K(\epsilon, \alpha_s)$ is contained in the residue of the simple pole, $K_1(\alpha_s)$, which is completely determined by the anomalous dimension γ_K, and in turn determines the coefficients of all higher poles. One can approximate the exact recursion relation by truncating the four–dimensional β function $\hat{\beta}(\alpha_s)$ to some chosen order.

It is useful to collect the terms in K corresponding to leading poles $((\alpha_s/\epsilon)^n)$, next–to–leading poles $(\alpha_s^{n+1}/\epsilon^n)$, and so on. This is done by writing

$$
\begin{aligned}
K(\epsilon, \alpha_s) &= \sum_{m=0}^{\infty} \mathcal{K}_m(\epsilon, \alpha_s) \,, \\
\mathcal{K}_m(\epsilon, \alpha_s) &= \sum_{n=1}^{\infty} K_n^{(n+m)} \left(\frac{\alpha_s}{\pi}\right)^{n+m} \frac{1}{\epsilon^n} \,.
\end{aligned}
\tag{3.4}
$$

According to general arguments, one expects the leading poles in K to be completely determined to all orders by the one–loop anomalous dimension and the one–loop β

[2]Notice that $K(\epsilon, \alpha_s)$ has only one pole per loop. The leading pole in the exponent of $\Gamma(Q^2)$, $(\alpha_s^n/\epsilon^{n+1})$, is generated at each order by the integration of the anomalous dimension γ_K.

function; similarly, the m–th tower of poles, $\mathcal{K}_m$ is expected to require the anomalous dimension and the β function to $m+1$ loops.

In the remainder of this section, the following facts about these towers of poles will be established:

- the recursion relation for the perturbative coefficients $K_n^{(m)}$ can be explicitly solved including *all* orders in the β function.
- All the resulting series of poles, $\mathcal{K}_m(\epsilon, \alpha_s)$, can be summed.
- Upon summation, $\mathcal{K}_m(\epsilon, \alpha_s)$ is an analytic function of α_s and ϵ, *regular* as $\epsilon \to 0$ for $m > 0$. The only singularity at $\epsilon \to 0$ is logarithmic and completely determined by a one loop calculation.
- The finite limits $\mathcal{K}_m(0, \alpha_s)$ $(m > 0)$ can be computed for *all* m in terms of the perturbative coefficients of β and γ_K. They reconstitute a power series in α_s.

As with most recursion relations, it is useful to start with the simplest cases.

3.1 One loop

The first of Eqs. (3.3) is readily solved and yields

$$K_1^{(m)} = \frac{1}{2m} \gamma_K^{(m)} \; . \tag{3.5}$$

In the second of Eqs. (3.3) one can start by truncating the β function at $\mathcal{O}(\alpha_s^2)$, keeping only the one–loop coefficient b_0. Using Eq. (3.5) one readily finds an expression for the generic coefficient $K_n^{(m)}$,

$$K_n^{(m)} = \frac{1}{2m} \left(- \frac{b_0}{4} \right)^{n-1} \gamma_K^{(m-n+1)} \; , \tag{3.6}$$

which, as we will verify, is exact for $n = m$. Then

$$\mathcal{K}_0(\epsilon, \alpha_s) = \sum_{n=1}^{\infty} K_n^{(n)} \left(\frac{\alpha_s}{\pi \epsilon} \right)^n = \frac{2 \gamma_K^{(1)}}{b_0} \ln \left(1 + \frac{b_0 \alpha_s}{4 \pi \epsilon} \right) \; . \tag{3.7}$$

Notice that this implies that the resummation of all the leading poles in the ratio (2.11) degrades the singularity to a logarithm:

$$\left. \frac{\Gamma\left(1, \alpha_s(Q^2), \epsilon\right)}{\Gamma\left(-1, \alpha_s(Q^2), \epsilon\right)} \right|_{L.P.} = \left(1 + \frac{b_0 \alpha_s}{4 \pi \epsilon} \right)^{\mathrm{i} \pi \frac{\gamma_K^{(1)}}{b_0}} \; . \tag{3.8}$$

Since all other poles are perturbatively weaker, one might expect that they should resum to even weaker singularities, or to regular functions. This in fact turns out to be the case.

3.2 Two loops

To resum next–to–leading poles, one must include the next coefficient of the β function, b_1, as well as $\gamma_K^{(2)}$. Eq. (3.3) may still be solved, and gives

$$K_n^{(m)} = \frac{1}{2m}\left[\sum_{k=0}^{n-1}\begin{pmatrix} n-1 \\ k \end{pmatrix}\left(-\frac{b_0}{4}\right)^{n-1-k}\left(-\frac{b_1}{4}\right)^{k}\gamma_K^{(m-n-k+1)}\right] , \tag{3.9}$$

where the sum is further truncated by the requirement that $\gamma_K^{(p)} = 0$ for $p < 1$. One recovers the previous answer, Eq. (3.7), for the leading poles. Next–to–leading poles are given by

$$K_{n-1}^{(n)} = \frac{1}{2n}\left(-\frac{b_0}{4}\right)^{n-3}\left[-\frac{b_0}{4}\gamma_K^{(2)} - (n-2)\frac{b_1}{4}\gamma_K^{(1)}\right] , \tag{3.10}$$

Again, the corresponding series is summable, and yields

$$\begin{aligned} \mathcal{K}_1(\epsilon,\alpha_s) &= \sum_{n=1}^{\infty} K_n^{(n+1)}\left(\frac{\alpha_s}{\pi}\right)^{n+1}\frac{1}{\epsilon^n} \\ &= \frac{8\epsilon}{b_0^2}\left(\frac{2\gamma_K^{(1)}b_1}{b_0} - \gamma_K^{(2)}\right)\ln\left(1+\frac{b_0\alpha_s}{4\pi\epsilon}\right) \\ &- \frac{\alpha_s}{\pi}\left[\frac{2}{b_0}\left(\frac{\gamma_K^{(1)}b_1}{b_0} - \gamma_K^{(2)}\right) + \frac{2\gamma_K^{(1)}b_1}{b_0^2}\frac{4\pi\epsilon}{b_0\alpha_s+4\pi\epsilon}\right] . \end{aligned} \tag{3.11}$$

The logarithmic singularity receives a new contribution, which, however, is supressed by a power of ϵ. Thus $\mathcal{K}_1$ has a finite limit as $\epsilon \to 0$, given by

$$\mathcal{K}_1(0,\alpha_s) = \frac{2\alpha_s}{\pi b_0}\left[\gamma_K^{(2)} - \frac{\gamma_K^{(1)}b_1}{b_0}\right] . \tag{3.12}$$

3.3 All loops

Encouraged by the simple form of the answers obtained for $\mathcal{K}_0$ and $\mathcal{K}_1$, one sets out to include higher orders in $\hat{\beta}(\alpha_s)$. It turns out that the recursion relation Eq. (3.3)

is solvable to all orders, and the solution is a straightforward generalization of Eqs. (3.6) and (3.9). One finds

$$K_n^{(m)} = \frac{1}{2m} \sum_{p_1,\ldots,p_{n-1}=0}^{\infty} \left[\prod_{j=1}^{n-1} \left(-\frac{b_{p_j}}{4} \right) \right] \gamma_K^{(m-n+1-\sum_{i=1}^{n-1} p_i)} \, . \tag{3.13}$$

To identify explicitly the towers of subleading poles, it is useful to reorganize the sum isolating the perturbative coefficients of the anomalous dimension γ_K. This is done by writing Eq. (3.13) as

$$K_n^{(m)} = \frac{1}{2m} \sum_{q_1=0}^{\infty} \gamma_K^{(m-n+1-q_1)} \left\{ \sum_{q_2=0}^{q_1} \cdots \sum_{q_{n-1}=0}^{q_{n-2}} \left[\prod_{j=1}^{n-1} \left(-\frac{b_{q_j - q_{j+1}}}{4} \right) \right] \right\} \, , \tag{3.14}$$

where the sum over q_1 is effectively cut off by the fact that for $q_1 > m - n + 1$ the summand vanishes. It is not trivial to reconstruct the general form of the m-th tower of subleading poles from Eq. (3.14); however, upon generating a sufficient number of examples, one recognizes that

$$K_{n-m}^{(n)} = \frac{1}{2n} \left(-\frac{b_0}{4} \right)^{n-m-1} \sum_{p=1}^{m+1} C_p(n,m) \; \gamma_K^{(m+2-p)} \, , \tag{3.15}$$

where

$$C_p(n,m) = \sum_{k=0}^{p-1} \binom{n-m-1}{k} \left(-\frac{b_0}{4} \right)^{-k} \mathcal{C}_k^{(p)}(b_1, \ldots, b_{p-1}) \, . \tag{3.16}$$

The coefficients $\mathcal{C}_k^{(p)}(b_1, \ldots, b_{p-1})$ are polynomial expressions in the b_i's, given by

$$\mathcal{C}_k^{(p)}(b_1, \ldots, b_{p-1}) = \sum_{\substack{m_i = 1 \\ \sum_{i=1}^k m_i = p-1}}^{p-1} \left[\prod_{i=1}^{k} \left(-\frac{b_{m_i}}{4} \right) \right] \, . \tag{3.17}$$

The key observation concerning Eqs. (3.15)–(3.17) is that the dependence on n is simple, so that summation over n can explicitly be performed for all m. In fact, for fixed m, Eq. (3.15) can be rewritten as

$$K_{n-m}^{(n)} = \frac{1}{2n} \left(-\frac{b_0}{4} \right)^{n-m-1} \sum_{r=0}^{m} \mathcal{D}_r(m) n^r \, , \tag{3.18}$$

with easily determined coefficients $\mathcal{D}_r(m)$. All the series to be summed are thus simple power series, in fact derivatives of the logarithmic series which resums the

leading poles, up to subtractions of a finite number of terms. It is easy to verify that the results obtained for $\mathcal{K}_m(\epsilon, \alpha_s)$ are finite in the limit $\epsilon \to 0$. One can in fact determine the form of this limit explicitly for all m. It is given by polinomial expressions in the coefficients of the β function, similar to Eqs. (3.15)–(3.17):

$$\mathcal{K}_m(0, \alpha_s) = \frac{1}{m} \left(\frac{\alpha_s}{\pi} \right)^m \frac{2}{b_0} \sum_{p=0}^{m} B_p(m) \; \gamma_K^{(m+1-p)} \; , \tag{3.19}$$

where

$$B_p(m) = \sum_{j=0}^{p} \mathcal{B}_j(m,p) \left(- \frac{b_0}{2} \right)^{-j} \; , \tag{3.20}$$

and

$$\mathcal{B}_j(m,p) = \sum_{\substack{q_i = 1 \\ \sum_{i=1}^{j} q_i = p}}^{m} \left[\prod_{i=1}^{j} \left(\frac{b_{q_i}}{2} \right) \right] \; . \tag{3.21}$$

The most amusing fact here is perhaps that in the limit $\epsilon \to 0$ one recovers a *finite* perturbative expansion in powers of α_s/π.

The conclusion of this exercise is that the counterterm function $K(\epsilon, \alpha_s)$ is completely determined, as an analytic function of ϵ, α_s, and the perturbative coefficients of β and γ_K. In the limit $\epsilon \to 0$ this function has a logarithmic singularity, arising from the resummation of the leading poles, and a contribution which is independent of ϵ and can be computed as a perturbative expansion in powers of α_s/π, with coefficients determined again by the functions β and γ_K. Explicitly

$$K(\epsilon, \alpha_s) = K_{DIV}(\epsilon, \alpha_s) + K_{FIN}(\alpha_s) + \mathcal{O}(\epsilon, \epsilon \ln \epsilon) \; , \tag{3.22}$$

with $K_{DIV}(\epsilon, \alpha_s)$ given by Eq. (3.7), and

$$\begin{aligned} K_{FIN}(\alpha_s) &= \sum_{m=1}^{\infty} \mathcal{K}_m(0, \alpha_s) = \frac{2}{b_0} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\alpha_s}{\pi} \right)^m \mathcal{A}_m \\ \mathcal{A}_m &= \sum_{p=0}^{m} B_p(m) \; \gamma_K^{(m+1-p)} \; . \end{aligned} \tag{3.23}$$

None of the series involved in this computation shows signs of factorial growth in the coefficients. In the final expression, Eq. (3.23), the growth of the coefficients is implicit in their dependence on b_i and $\gamma_K^{(i)}$, the series in Eq. (3.23) being otherwise convergent with a logarithmic behavior.

4 One–loop resummation

The success in the computation of the counterterm function $K(\epsilon, \alpha_s)$ is encouraging in view of a more explicit evaluation of the full form factor. It is particularly tempting to study the behavior of the integrals in Eq. (2.10) in the small–ξ^2 region, where one expects to find obstructions due to the Landau pole, which should however cancel in the ratio of the timelike to the spacelike factor, Eq. (2.11), which is ultraviolet dominated.

A natural place to start is a more precise characterization of the Landau pole in the context of dimensional regularization. The one–loop running coupling for finite ϵ can still be characterized by a mass scale, defined as the value of the renormalization scale where the coupling diverges, just as in conventional dimensional transmutation. In fact, Eq. (2.7) still has a simple pole located at

$$\mu^2 = \Lambda^2 \equiv Q^2 \left(1 + \frac{4\pi\epsilon}{b_0 \alpha_s(Q^2)} \right)^{-1/\epsilon} , \tag{4.1}$$

and one can use the scale Λ^2 to define the coupling, as

$$\alpha_s(Q^2) = \frac{4\pi\epsilon}{b_0 \left[\left(\frac{Q^2}{\Lambda^2} \right)^{\epsilon} - 1 \right]} . \tag{4.2}$$

By inspection, Eqs. (2.7), (4.1) and (4.2) reduce to their four–dimensional counterparts as $\epsilon \to 0$. There are however two important observations to be made.

- The running coupling defined by Eq. (4.2) vanishes as $Q^2 \to 0$ for fixed ϵ, provided **Re** $\epsilon < 0$, in agreement with Eq. (2.7) and with Eq. (2.9). This behavior for small Q^2 is simply due to dimensional counting: in fact, since we are keeping the regulator in place, we are studying the renormalized coupling $g_R = g_0 \mu^{-\epsilon}$ for fixed bare coupling, in $d > 4$, where it vanishes as a power of the scale.

- For general (complex) ϵ, the Landau pole in Eq. (4.1) is located at complex values of the scale μ^2. In particular, for real negative ϵ, the location of the pole has a nonvanishing imaginary part provided $\epsilon < -b_0 \alpha_s(Q^2)/(4\pi)$, $\epsilon \neq -1/n$. Thus, in general, the Landau pole in the dimensionally regulated exponent of Eq. (2.10) is *not* on the integration contour. One expects, and I will verify below, that the integrals appearing in Eq. (2.10) may be evaluated explicitly

in terms of analytic functions of α_s and ϵ. When ϵ becomes close to 0, the pole migrates to the integration contour, and correspondingly the integral will develop a cut. Nonperturbative information (if any) will be encoded in the structure of the singularities at the cut, and in the corresponding higher order generalizations.

Armed with these obervations, one can proceed to verify the integrability of $\log \Gamma(Q^2)$, starting with the simplest situation, in which only one–loop information is retained. The integration can be performed using three different methods, which highlight different features of the exponent.

4.1 Integration by series

The idea of re–expanding the one–loop running coupling, Eq. (2.7), in powers of $\alpha_s(Q^2)$ is perhaps the first that comes to mind, in view of its frequent usage in the context of the study of power corrections. One writes

$$\overline{\alpha}\left(\frac{\mu^2}{\mu_0^2}, \alpha_s(\mu_0^2), \epsilon\right) = \left(\frac{\mu_0^2}{\mu^2}\right)^\epsilon \alpha_s(\mu_0^2) \sum_{n=0}^\infty \left[\frac{1}{\epsilon}\left(\left(\frac{\mu_0^2}{\mu^2}\right)^\epsilon - 1\right)\frac{b_0}{4\pi}\alpha_s(\mu_0^2)\right]^n , \qquad (4.3)$$

and one integrates the series term by term, looking for possible factorial growth in the coefficients of the series obtained after integration. This would then be interpreted by standard methods as a sign of nonperturbative contributions. In the present case, changing variables in each integral in Eq. (2.10) as

$$\lambda^2 \to z = \left(\frac{\mu^2}{\lambda^2}\right)^\epsilon - 1 , \qquad (4.4)$$

one finds that all integrals as easily performed. The integration of the anomalous dimension γ_K generates terms that are indipendent of ξ, the integration variable of the outer integral in Eq. (2.10). These terms are divergent at the lower limit of integration, $\xi^2 = 0$, but they are cancelled by the ξ–independent terms in the expansion of the counterterm function $K(\epsilon, \alpha_s)$, a fact that can be explicitly verified to all orders by using the results of Section 3. Considering for simplicity the spacelike form factor (obtained from Eq. (2.10) by changing $Q^2 \to -Q^2$), the result is then

$$\log \Gamma\left(\frac{-Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \frac{2}{b_0}\, a(\mu^2) \sum_{n=0}^\infty \left\{\left(\frac{a(\mu^2)}{\epsilon}\right)^n \frac{(-1)^{n+1}}{n+1}\right. \qquad (4.5)$$

$$\times \left[-\frac{\gamma_K^{(1)}}{2\epsilon^2} \sum_{k=0}^{n} \frac{1}{k+1} \left(\left(1 - \left(\frac{\mu^2}{Q^2} \right)^{\epsilon} \right)^{k+1} - 1 \right) \right.$$
$$\left. \left. - \frac{G^{(1)}(\epsilon)}{\epsilon} \left(\left(1 - \left(\frac{\mu^2}{Q^2} \right)^{\epsilon} \right)^{n+1} - 1 \right) \right] \right\} ,$$

where I defined

$$a(\mu^2) = \frac{b_0}{4\pi} \alpha_s(\mu^2) , \tag{4.6}$$

and I wrote the function G perturbatively as

$$G(-1, \overline{\alpha}, \epsilon) = \sum_{n=1}^{\infty} G^{(n)}(\epsilon) \left(\frac{\overline{\alpha}}{\pi} \right)^n , \tag{4.7}$$

with $G^{(1)}(\epsilon) = C_F(3 - \epsilon(\zeta(2) - 8))/2 + \mathcal{O}(\epsilon^2)$.

It is apparent that the series in Eq. (4.5) is far from pathological. It does not exhibit any factorial growth, it is alternating in sign, and definitely has a finite radius of convergence. In fact, one recognizes a logarithmic series in the terms with a single sum, whereas the terms in the nested sum are finite harmonic sums of the kind encountered performing Mellin transforms of Altarelli–Parisi kernels, which are related to polylogarithms.

Having found that the series arising from the integration of Eq. (2.10) is well–behaved, clearly the simplest method to sum it is to perform the integrals directly, using Eq. (2.7) instead of the Taylor expansion in Eq. (4.3).

4.2 Analytic expression

Direct integration proceeds through steps that are closely related to the ones outlined in the previous subsection. Through the change of variable in Eq. (4.4) one trivially integrates the one–loop anomalous dimension, obtaining a logarithm. The logarithm is not integrable over the scale ξ^2 because of the singularity at $\xi^2 = 0$, which is however subtracted by the contribution of the counterterm function (here one must use for $K(\epsilon, \alpha_s)$ the resummation of the leading poles, Eq. (3.7)). Using once more the same change of variables, the second integration can also be performed. After minor reshuffling with dilogarithm identities, and inserting the values of $\gamma_K^{(1)}$ and $G^{(1)}$, the result reads

$$\log \Gamma \left(\frac{-Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = -\frac{2C_F}{b_0} \left\{ \frac{1}{\epsilon} \, \mathrm{Li}_2 \left[\left(\frac{\mu^2}{Q^2} \right)^{\epsilon} \frac{a(\mu^2)}{a(\mu^2) + \epsilon} \right] \right. \tag{4.8}$$
$$\left. - \, C(\epsilon) \, \ln \left[1 - \left(\frac{\mu^2}{Q^2} \right)^{\epsilon} \frac{a(\mu^2)}{a(\mu^2) + \epsilon} \right] \right\} ,$$

where $C(\epsilon) = G^{(1)}(\epsilon)/C_F$. Eq. (4.8) sums the series in Eq. (4.5), and has several rather remarkable properties. First, since the form factor is RG invariant, and I used the exact solution of the one–loop RG equation for $\overline{\alpha}$ to perform the integration, Eq. (4.8) should be independent of μ^2 to one–loop accuracy in $\beta(\alpha_s)$. This is easily verified, and one obtains

$$\begin{aligned}
\log \Gamma\left(\frac{-Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) &= \log \Gamma\left(-1, \alpha_s(Q^2), \epsilon\right) \qquad (4.9) \\
&= -\frac{2C_F}{b_0}\left\{\frac{1}{\epsilon}\,\mathrm{Li}_2\left[\frac{a(Q^2)}{a(Q^2)+\epsilon}\right] + C(\epsilon)\,\log\left[1 + \frac{a(Q^2)}{\epsilon}\right]\right\} .
\end{aligned}$$

Eq. (4.9), and its generalization including two–loop effects given in Section 5, are the central results of the present paper. The quark form factor is compactly expressed in terms of a simple analytic function of the coupling and of ϵ, which is manifestly independent of the renormalization scale, and can be re–expanded in powers of $\alpha_s(Q^2)$ to generate perturbative results that can be directly compared with Feynman diagram calculations. In particular, Eq. (4.9) resums all leading $(\alpha_s^n/\epsilon^{2n})$ and next–to–leading $(\alpha_s^n/\epsilon^{2n-1})$ infrared and collinear poles in the form factor.

The second important feature of Eqs. (4.8) and (4.9) is that it is possible to study the structure of the singularities of the form factor in the complex ϵ plane. A detailed analysis of this question is left for future work. Here I will simply note a few aspects that are apparent in Eq. (4.9). First, the Landau singularity, as expected, appears in Eq. (4.9) as one of the two branching points of a cut which is shared by both terms. The cut can be taken to run along the negative real axis in the ϵ plane (the "physical" region), from $\epsilon = -a(Q^2)$ (the Landau singularity, see Eq. (4.1)) to $\epsilon = 0$. At the Landau point, both terms in Eq. (4.9) diverge logarithmically. On the other hand, the limit $\epsilon \to 0$, which is the physically relevant one, can be studied explicitly; one finds

$$\begin{aligned}
\log \Gamma\left(-1, \alpha_s(Q^2), \epsilon\right) &= \frac{2C_F}{b_0}\left[-\frac{\zeta(2)}{\epsilon} + \frac{1}{a(Q^2)} + \left(\frac{1}{a(Q^2)} - \frac{3}{2}\right)\log\left(\frac{a(Q^2)}{\epsilon}\right)\right. \\
&+ \left.\mathcal{O}(\epsilon, \epsilon\log\epsilon)\right] . \qquad (4.10)
\end{aligned}$$

Eq. (4.10) has interesting features: one observes that resumming all the leading poles in the logarithm of the form factor softens the singularity in a manner similar to what happens for the counterterm function (see Eq. (3.7)). In this case the leading singularity is a simple pole, with a residue which does *not* depend on the

coupling, an thus not on the scale either. Including higher loops in β corresponds to resumming weaker singularities, thus one might expect that it would not affect the leading singularity in Eq. (4.10). In Section 5, I will verify that this is indeed the case at two loops. Another interesting feature of Eq. (4.10) is the appearance of a term in the form factor of the form $\exp(c/a(Q^2))$, which, in the $\epsilon \to 0$ limit, translates into a power behavior of the form $(Q^2/\Lambda^2)^c$. The term in question in the present case is not of direct physical significance: it leads to a *positive* fractional power of Q^2, and it is tightly connected with the infrared divergence; in fact, as we will see, it cancels in the ratio (2.11), which is the simplest quantity of physical interest that can be built out of the form factor. It is however interesting that such a term can arise in this way: one sees that dimensional regularization and resummation can combine to give well–defined (and gauge–invariant) power "corrections", which might be of physical interest if the present method can be generalized to construct infrared safe quantities to the same level of accuracy.

As a final test of the correctness of Eq. (4.9), one can compute the ratio $\Gamma(Q^2)/\Gamma(-Q^2)$ in the limit $\epsilon \to 0$. To the present accuracy, one finds

$$\begin{aligned} \log\left[\frac{\Gamma\left(1, \alpha_s(Q^2), \epsilon\right)}{\Gamma\left(-1, \alpha_s(Q^2), \epsilon\right)}\right] &= \frac{2C_F}{b_0}\left[\, \mathrm{i}\pi\left(1 + \log\frac{a(Q^2)}{\epsilon}\right)\right. \\ &\left. - \left(\frac{1}{a(Q^2)} - \frac{3}{2} + \mathrm{i}\pi\right)\log\left(1 + \mathrm{i}\pi a(Q^2)\right) + \mathcal{O}(\epsilon)\right] , \end{aligned} \tag{4.11}$$

which agrees with Eq. (3.7) for the logarithmically divergent term, and agrees with the results of [7] for the finite contribution, which was computed there to this same accuracy using Eq. (2.11). Notice that the the term responsible for the power behavior in Q^2 has cancelled together with the simple pole.

4.3 Integration over the coupling

Generalizing Eqs. (4.8) and (4.9) to include two– and higher loop effects would appear to be nontrivial with the techniques described so far. In fact, even at two loops the RG equation for $\overline{\alpha}$ cannot be solved in terms of elementary functions without approximations. There is, fortunately, no need to do that: one can change variables according to

$$\frac{d\mu}{\mu} = \frac{d\alpha_s}{\beta(\alpha_s)} , \tag{4.12}$$

i.e. use the coupling directly as integration variable, without the need to introduce an explicit expression for $\alpha_s(\mu^2)$ in the intermediate stages. At one loop

$$\frac{d\mu^2}{\mu^2} = -\frac{d\alpha_s}{\alpha_s}\frac{1}{\epsilon + \frac{b_0}{4\pi}\alpha_s} \ . \tag{4.13}$$

One recognizes yet another way to characterize the Landau singularity in dimensional regularization: it arises because the one–loop β function with $\epsilon \neq 0$ does not have a (double) zero at $\alpha_s = 0$, as usual, but has two distinct zeroes, one at the origin and one located at $\alpha_s = -4\pi\epsilon/b_0$. This second zero will be on the integration contour for real negative ϵ, provided $|\epsilon| < b_0\alpha_s(Q^2)/(4\pi)$, as expected from the earlier calculations. For $\epsilon < 0$, this second zero, where the β function is decreasing, is the one responsible for asymptotic freedom, while at the origin in the α_s plane the β function vanishes with positive derivative, as it would in a QED-like theory, such as ϕ^4 in $d = 4$. In fact, this kind of behavior of dimensionally regularized theories (with the opposite sign of the β function) is familiar in statistical field theory [18], where dimensional continuation is used, for example, to study the properties of the (nontrivial) ϕ^4 theory in $d = 3$ starting from the (trivial) case $d = 4$. In the present case, one can see from the explicit solution, Eq. (2.7), that below the critical point $\epsilon = -b_0\alpha_s(Q^2)/(4\pi)$ the coupling decreases smoothly to 0, starting from the boundary value $\alpha_s(Q^2)$. Above the critical point, on the other hand (and thus getting closer to the physical value $\epsilon = 0$), the coupling develops a Landau pole, so that it becomes impossible to evolve smoothly along real values of the scale.

Inplementing the change of variables Eq. (4.12), one sees that the only remaining dependence on the scale μ^2 is in the upper limit of integration of the anomalous dimension integral, which is now $\alpha_s(\mu^2)$, and in the counterterm function $K(\epsilon, \alpha_s(\mu^2))$. These two dipendences must (and do) cancel by RG invariance, in the same way in which the singularities at $\xi^2 \to 0$ cancelled in the previous versions of the calculation. At the lower limit of integration, $\xi^2 = 0$, one must use $\alpha_s(0) = 0$, in accordance with the arguments given above and at the beginning of Section 4; one then recovers directly Eq. (4.9). This third procedure of integration is clearly, and by far, the most straightforward; furthermore, it is readily generalizable to to more than one loop, as I will now show.

5 Two–loop resummation

The generalization of the procedure outlined in Section 4.3 to include higher orders in the β function is straightforward. At l loops, $\beta(\alpha_s)$ is a polynomial of degree $l+1$, thus Eq. (2.10) will be expressed in terms of a double integral of a combination of rational function of α_s, which in general will be computable by partial fractioning in terms of combinations of polylogarithms. The result will be expressed in terms of the perturbative coefficients of the functions $\beta(\alpha_s)$, $\gamma_K(\alpha_s)$ and $G(1, \overline{\alpha}, \epsilon)$, which are known to two loops; therefore I will briefly illustrate the procedure and the results in that case, where all the ingredients of the final expression are explicitly known, and the size remains manageable.

At two loops one must use, for each integral over the scale of the coupling in Eq. (2.10), the change of variables

$$\frac{d\mu^2}{\mu^2} = -\frac{d\alpha_s}{\alpha_s} \frac{1}{\epsilon + \frac{b_0}{4\pi}\alpha_s + \frac{b_1}{4\pi^2}\alpha_s^2} \ . \tag{5.1}$$

Partial fractioning of the denominators is achieved by introducing the two nontrivial zeroes of the two–loop β function, or, for convenience, the two zeroes of the equation $b_1\alpha^2 + b_0\alpha + 4\epsilon = 0$ (with $\alpha = \alpha_s/\pi$),

$$a_\pm = -\frac{b_0}{2b_1}\left(1 \pm \sqrt{1 - \frac{16\epsilon b_1}{b_0^2}}\right) \ . \tag{5.2}$$

Note that it will be possible to verify that the result reduces to the correct limit at one loop by using the fact that, as $b_1 \to 0$, a_- becomes the one–loop zero, $a_- \to -4\epsilon/b_0$, while a_+ diverges as $a_+ \to -b_0/b_1$.

The cancellation of the dependence on $\alpha_s(\mu^2)$ between the counterterm function K and the anomalous dimension integral takes place as usual. The final result can be written in the form

$$\begin{aligned}
\log\Gamma\left(-1, \alpha_s(Q^2), \epsilon\right) = & -\frac{2}{b_1(a_+ - a_-)}\left\{\left(G^{(1)}(\epsilon) + a_+ G^{(2)}(\epsilon)\right)\log\left(1 - \frac{\alpha_s(Q^2)}{\pi a_+}\right)\right. \\
& + \ 2\ \left(\gamma_K^{(1)} + a_+\gamma_K^{(2)}\right)\left[-\frac{1}{4\epsilon}\mathrm{Li}_2\left(\frac{\alpha_s(Q^2)}{\pi a_+}\right) + \frac{1}{2b_1 a_+(a_+ - a_-)}\log^2\left(1 - \frac{\alpha_s(Q^2)}{\pi a_+}\right)\right. \\
& - \frac{1}{b_1 a_-(a_+ - a_-)}\left(\mathrm{Li}_2\left(\frac{\pi a_+ - \alpha_s(Q^2)}{\pi(a_+ - a_-)}\right) - \mathrm{Li}_2\left(\frac{a_+}{a_+ - a_-}\right)\right. \\
& \left.\left.\left. + \log\left(1 - \frac{\alpha_s(Q^2)}{\pi a_+}\right)\log\left(\frac{\alpha_s(Q^2) - \pi a_-}{\pi(a_+ - a_-)}\right)\right)\right]\right\} \ + \ (a_+ \leftrightarrow a_-) \ .
\end{aligned} \tag{5.3}$$

It is fairly straightforward to check that Eq. (5.3) reduces to Eq. (4.9) when one takes $\{b_1, \gamma_K^{(2)}, G^{(2)}(\epsilon)\} \to 0$. On the other hand, the analytic structure of Eq. (5.3) is considerably more complicated than the one briefly discussed at one loop. In particular, the presence of a second nontrivial zero in $\beta(\alpha_s)$ intoduces a new branch point, so that the structure of cuts becomes more complicated. One does not expect, however, the behavior physical quantities related to the form factor to be qualitatively affected by the inclusion of two–loop effects, since these are understood to modify the running of the coupling by an amount which is a logarithmically suppressed and smooth funcion of the energy.

It is interesting to study the behavior of the two–loop resummed form factor, Eq. (5.3), in the neighborhood of the "physical" limit, $\epsilon \to 0$. One finds that, as expected, the simple pole in the logarithm of the form factor given in Eq. (4.10) is not affected by the inclusion of two–loop effects. The logarithmic singularities, however, are enhanced to $\log^2$ strength, and also the constant terms in the limit $\epsilon \to 0$ receive new nontrivial contributions. One finds

$$\begin{aligned}
\log \Gamma\left(-1, \alpha_s(Q^2), \epsilon\right) \; = \; & - \frac{\gamma_K^{(1)}}{b_0} \frac{\zeta(2)}{\epsilon} + \frac{2 b_1 \gamma_K^{(1)}}{b_0^3} \log^2 \left[\frac{4\pi\epsilon}{b_0 \alpha_s(Q^2)} \left(1 + \frac{\alpha_s(Q^2) b_1}{\pi b_0} \right) \right] \\
& + \frac{2}{b_0} \left(G_0^{(1)} + \frac{2\gamma_K^{(2)}}{b_0} - \frac{2\pi\gamma_K^{(1)}}{\alpha_s(Q^2) b_0} - \frac{2 b_1 \gamma_K^{(1)}}{b_0^2} \right) \log \left[\frac{4\pi\epsilon}{b_0 \alpha_s(Q^2)} \left(1 + \frac{\alpha_s(Q^2) b_1}{\pi b_0} \right) \right] \\
& + \frac{4\gamma_K^{(2)}}{b_0^2} \mathrm{Li}_2 \left(\frac{\alpha_s(Q^2) b_1}{\pi b_0 + \alpha_s(Q^2) b_1} \right) + \left(\frac{4\pi\gamma_K^{(2)}}{\alpha_s(Q^2) b_0 b_1} - \frac{2 G_0^{(2)}}{b_1} \right) \log \left(1 + \frac{\alpha_s(Q^2) b_1}{\pi b_0} \right) \\
& + \frac{4\pi\gamma_K^{(1)}}{\alpha_s(Q^2) b_0^2} + \left(1 - 2\zeta(2)\right) \frac{4 b_1 \gamma_K^{(1)}}{b_0^3} - \left(1 - \zeta(2)\right) \frac{4\gamma_K^{(2)}}{b_0^2} + \mathcal{O}\left(\epsilon, \epsilon \log \epsilon\right) \, ,
\end{aligned} \tag{5.4}$$

where $G_0^{(i)} = \lim_{\epsilon \to 0} G^{(i)}(\epsilon)$. Also in this limit, it is easy to check that taking $\{b_1, \gamma_K^{(2)}, G^{(2)}(\epsilon)\} \to 0$ one gets back Eq. (4.10).

Although it is not proved here, one can conjecture rather confidently that the simple pole in $\log \Gamma(Q^2)$ is completely determined at one loop, and receives no corrections from higher orders in β. The presence of the pole is in fact tightly connected with the one–loop zero of the β function, which is located at a distance $\mathcal{O}(\epsilon)$ from the origin. All higher–order zeroes have locations in the α_s plane that are independent of ϵ in the small ϵ limit, thus they are not expected to introduce further singular behavior in this region. Logarithmic singularities, on the other hand, arise both from the anomalous dimension integral and from the integration of the function G, thus one must expect that they get higher order corrections.

6 Summary and perspectives

I have discussed the quark form factor in the context of dimensionally regulated perturbative QCD. Building upon the results of Ref. [7], as well as upon the understanding developed previously by many authors [1]–[6], I have been able to derive resummed analytic expression for the form factor that are independent of the choice of renormalization scale, and can be systematically improved upon by including higher orders in the QCD β function.

The main ingredient in the derivation is the consistent usage of dimensional regularization as analytic continuation to general complex values of the space–time dimension, $d = 4 - 2\epsilon$. It turns out that this continuation does not only regulate ultraviolet (as well as infrared) singularities in Feynman integrals: it also regulates singularities such as the Landau pole, which arise upon resummation of classes of Feynman diagrams, and are usually interpreted as signals of nonperturbative physics. How this happens is well understood, and this understanding has been applied as a technical tool in statistical field theory for a long time: for $d > 4$, the QCD β function develops an asymptotically free fixed point at a distance $\mathcal{O}(\epsilon)$ from the origin in the α_s plane, while the origin itself becomes *infrared* free; the coupling vanishes like a power of the renormalization scale, and it does so smoothly for sufficiently large d; when d gets close to the physical value $d = 4$ the coupling develops a Landau pole for real values of the scale; this pole is then found on the integration contour of resummed expressions for QCD amplitudes, which, as a consequence, develop cuts with computable branching points and discontinuities.

The versatility of dimensional regularization was exploited here also to derive compact resummed analytic expressions for the counterterm function $K(\epsilon, \alpha_s)$: the fact that the renormalization group equation for a pure counterterm (with a finite anomalous dimension) turns into a recursion relation is well known; the fact that for the quark form factor the recursion relation is completely solvable to all orders in β, and the resulting towers of poles can be summed, provided a strong motivation to pursue the calculation for the complete form factor.

The calculations presented in this paper are not directly applicable to physical processes: the paper, after all, deals with a divergent quantity[3]. There are however several lines of enquiry that are opened by these results, and I believe are worth

[3]The modulus squared of Eq. (4.11) is an exception, as it enters directly the resummed Drell–Yan cross section; it was, however, computed already in Ref. [7]

pursuing.

First of all, it would be of great interest to extend results such as Eq. (4.9) and Eq. (5.3) to more general QCD amplitudes, in particular to amplitudes with more than two colored particles. This is not straightforward, because more complicated amplitudes have a nontrivial tensor structure in color space, and evolution equations like Eq. (2.2) correspondingly turn into matrix equations. It should be emphasized however that much of the necessary technology has already been developed in the context of the resummation of large–x logarithms for the production of heavy quark pairs and other colored final states in hadronic collisions [10]. In that case one resums large corrections due to the presence of + distributions at each order in perturbation theory, arising from the imperfect cancellation of real and virtual contributions. In the present case one would be interested in purely virtual corrections contributing to the cross section with terms proportional to δ functions. I believe that a generalization of the present results in this direction should be possible, and work to achieve it is under way.

A more ambitious goal would be to attempt the construction of *finite*, physical quantities to the same accuracy achieved here for the form factor. This would require controlling the infrared/collinear singularities of dimensionally regulated real emission amplitudes to all orders. While this is not likely to be possible in a general multi–scale process, progress might conceivably be made in the simplest cases, such as the inclusive Drell-Yan cross section, by introducing suitable approximations to the d–dimensional cross section.

On a more speculative note, one may wonder whether the analytic structure uncovered in the present paper may shed some light on nonperturbative features of QCD amplitudes. It was already noticed in the comments to Eq. (4.10) that in the neighborhood of $\epsilon = 0$ one finds terms that translate into powers of the ratio Q^2/Λ^2. More generally, the analytic features of the form factor near the Landau cut might provide suggestions to build models for the nonperturbative behavior of simple QCD amplitudes, as well as for the behavior of the coupling itself, much as renormalon calculations did [12, 19].

In the context of finite order perturbative QCD calculations, the existence of genuinely RG–invariant expressions such as Eqs. (4.9) and (5.3) might prove useful as a model for the renormalization scale dependence of more complicated QCD amplitudes. Not surprisingly, the results for the form factor are naturally expressed

in terms of $\alpha_s(Q^2)$, since Q^2 is the only perturbative scale in the amplitude. One can however go back to equations such as (4.8), choose a different renormalization scale, say $\mu = Q/2$, re–expand the form factor to a finite perturbative order, and study the variations of the answer with μ.

Finally, it should be mentioned that large corrections to perturbative QCD amplitudes due to analytic continuation (the "π^2 terms") do not arise only in quark–initiated process. A case in point is Higgs production via gluon fusion, in an effective theory in which the top quark has been integrated out [20]. Also in that case the present techniques may well prove useful and provide a very accurate and rigorous evaluation of the impact of these terms to all orders in the coupling.

In summary, I believe that the results presented here should be of interest for the study of several aspects, both formal and practical, of perturbative QCD. Much work remains to be done.

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